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Reconstructing space–time from world-lines of charged particles

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Abstract

It is possible to reconstruct a given space–time (M, g) furnished with a 2-form Ω (the electromagnetic strength) from the 6-dimensional manifold parametrizing the world-lines in M of the particles with rest-mass m_0 (or 0) and charge $\pm e_0$ (or 0).

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0. Introduction

The paper [C] shows how to reconstruct a space–time (M, g) (points and metric tensor) from the manifold E of "geodesics" of M; the present article extends that result and shows how it is possible to reconstruct a given space–time (M, g) furnished with a 2-form Ω (the electromagnetic strength) from the manifold of the world-lines in M of the particles with rest-mass m_0 (or 0) and charge $\pm e_0$ (or 0).

Given a complexified space–time (M, g) furnished with a holomorphic 2-form Ω coming from the curvature of a Kaluza–Klein holomorphic principal $\mathbb{C}^*$-bundle $\pi : P \to M$ supplied with a holomorphic connection ω, the world-lines in M of the particles with rest-mass and charge $(m_0, \pm e_0)$ or $(0, 0)$, in general, make a manifold E of (complex) dimension 6 where the world-lines of particles with zero rest-mass make a hypersurface E_0 of dimension 5; on the other hand given a suitable metric to P, it is possible to "lift" every world-line in M to a null geodesic of P defined up to the action of $\mathbb{C}^*$.

Therefore the space N of null geodesics of P is, in general, a manifold of dimension 7 such that $N/\mathbb{C}^* = E$ and, in the favorable cases, the projection map $\epsilon : N \to E$ sending null geodesics of P to world-lines of M makes N a principal $\mathbb{C}^*$-bundle on E.

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Starting with the bundle $N \to E$, it is possible to reconstruct (M, g) and (P, ω) (and then Ω) by exploiting only the complex manifold structures.

In addition the world-lines in M are reobtained as curves defined in a natural way by E and N; this is possible because there is a contact structure on E. This structure does not intervene in the reconstruction of $\pi : (P, \omega) \to (M, g)$ but simply it guarantees that deformations due to "torsion terms" will not get into in the reconstruction of the world-lines.

The last section of this paper will analyze under which conditions there can be a contact structure on such a bundle and will furnish a necessary and sufficient condition in the presence on $E$of a meromorphic symplectic form with poles of a well defined type on E_0; this symplectic form, moreover, will be shown to verify Weil's integrality conditions.

1. From space–time to the space of world-lines

Definition 1.1. A *Kaluza–Klein structure* is a 5-tuple (P, π, M, g, ω) where M is a complex manifold, g is a holomorphic Riemannian metric on M, the map $\pi : P \to M$ makes P a principal holomorphic $\mathbb{C}^*$-bundle on M and ω is a holomorphic connection 1-form on P.

The action of $\mathbb{C}^*$ on P defines a fundamental holomorphic vector field A such that $\pi_*(A) = 0$ and $\omega(A) = 1$. It will be denoted by Ω, the curvature holomorphic 2-form of the structure (defined locally as $\Omega = \partial\sigma^*\omega$ for any local section $\sigma : U \to P$) and (M, g) will be interpreted as a complexified space–time with Ω as an electromagnetic field strength on M.

With respect to the structure (M, g, Ω) we will consider only particles with "mass and charge" in the "triplet" $(m_0, -e_0)$, $(0, 0)$ and (m_0, e_0) (that is of the form $(\sigma^2 \cdot m_0, \sigma \cdot e_0)$ where m_0 and e_0 are fixed constant numbers in $\mathbb{C}^*$ and $\sigma \in \{-1, 0, 1\}$).

Definition 1.2. A *world-line of a particle in the triplet* is a couple (σ, α) of a "sign" $\sigma \in \{-1, 0, 1\}$ and a regular holomorphic map $\alpha : D \to M$ defined on a simply connected open region D of $\mathbb{C}$ such that

$$\nabla_{\dot\alpha}\dot\alpha = \sigma \cdot e_0 \cdot (\uparrow \Omega \rfloor \dot\alpha) \quad \text{and} \quad g(\dot\alpha, \dot\alpha) = \sigma^2 \cdot m_0^2.$$

In the following will be fixed the constant $k = m_0/e_0$.

On the manifold P we will consider the non-degenerate holomorphic Riemannian metric g' defined by

$$g'(X', Y') = g(\pi_*(X'), \pi_*(Y')) - k^2 \cdot \omega(X') \cdot \omega(Y'),$$

this metric is invariant by the action of $\mathbb{C}^*$ and verifies the equality: $g'(A, A) = -k^2$.

Theorem 1.3. *For every world-line $(\sigma, \alpha) : D \to M$ there exists a null geodesic γ in P such that $\pi \circ \gamma = \alpha$ and $\omega(\dot\gamma) = \sigma \cdot e_0$. The null geodesic γ is defined up to the action of $\mathbb{C}^*$. Conversely for every null geodesic $\gamma : D \to P$ the map $\pi \circ \gamma : D \to M$ with a suitable changement of the affine parameter is a world-line of a particle in the triplet.*

Proof. Since the problem is local we choose a trivializing open subset U of M with coordinates z^j on U and a coordinate w on $\mathbb{C}^*$, such that $\partial_w = A$. The connection form expressed as $\omega = \omega_j \cdot \partial z^j + \partial w$, we have to look for a holomorphic function $\beta : D \to \mathbb{C}$ such that the curve $\gamma(t) = (\alpha(t), \beta(t))$ is null and stationary with respect to the functional defined by the "arc-length" $\mathcal{L} : T'P \to \mathbb{C}$ given by $\mathcal{L}(X') = g'(X', X') = \mathcal{L}(z, w, Z, W) = g_{ab}(z) \cdot Z^a \cdot Z^b - k^2 \cdot [\omega_a(z) \cdot Z^a + W]^2$, that is verifying the two Lagrange equations

(1) $\omega \rfloor \dot{\gamma} = \text{constant} = Q$,

(2) $\nabla_{\dot{\alpha}} \dot{\alpha} = Q \cdot (\uparrow \Omega \rfloor \dot{\alpha})$,

and moreover the nullity condition

(3) $g(\dot{\alpha}, \dot{\alpha}) - k^2 \cdot Q^2 = 0$.

Since α is the world-line of a particle in the triplet it is enough to take a holomorphic function $\beta : D \to \mathbb{C}$ such that $\dot{\beta} = \sigma \cdot e_0 - \omega_j \cdot \dot{\alpha}^j$ and consider $Q = \sigma \cdot e_0$. □

Definition 1.4. The *space of signed tangent vectors of* (M, g) is the regular hypersurface $\mathbb{S}(T'M) = \{[Y, h] \in \mathbb{P}(T'M \times \mathbb{C}) \colon g(Y, Y) = h^2\}$.

For every world-line (σ, α) it is well defined a map $\tilde{\alpha}_\sigma : D \to \mathbb{S}(T'M)$ by $\tilde{\alpha}_\sigma(z) = [\dot{\alpha}(z), \sigma \cdot m_0]$. The maps $\tilde{\alpha}_\sigma$ describe regular curves that foliate $\mathbb{S}(T'M)$.

If $(0, \alpha)$ is a null geodesic of M and α' is an affine reparametrization of α then $\tilde{\alpha}_0$ and $\tilde{\alpha}'_0$ run in the same leaf of $\mathbb{S}(T'M)$. Moreover if $\alpha'(z) = \alpha(z + u)$ then $\tilde{\alpha}_\sigma$ and $\tilde{\alpha}'_\sigma$ run in the same leaf and if $\alpha'(z) = \alpha(-z + u)$ then $\tilde{\alpha}_\sigma$ and $\tilde{\alpha}'_{-\sigma}$ run in the same leaf.

Conversely if $\tilde{\alpha}_\sigma$ and $\tilde{\alpha}'_\sigma$ run in the same leaf (with a common point) then one of the above three cases takes place.

Definition 1.5. The space of leaves of the maps $\tilde{\alpha}_\sigma$ in $\mathbb{S}(T'M)$ will be called the space $E(M) = E(M, g, \omega)$ *of the world-lines for the particles in the triplet.*

We will denote by $N(P)$ the space of null geodesics in (P, g') (cf. [L]), that is the space $\mathcal{Q}(P) = \{[X'] \in \mathbb{P}T'P \colon g'(X', X') = 0\}$ (the space of null directions of P) modulo the equivalence identifying two directions on a same null geodesic of P. This equivalence foliates $\mathcal{Q}(P)$ and $N(P)$ is exactly the space of the leaves of $\mathcal{Q}(P)$.

The projection of the null geodesics of P on the world-lines of M defines a surjective map $\epsilon : N(P) \to E(M)$ given by ϵ (leaf of $[X']$) = leaf of $[\pi_*(X'), k \cdot \omega(X')]$.

The action of $\mathbb{C}^*$ on P gives an action of $\mathbb{C}^*$ on $N(P)$ whose orbits are precisely the fibres of ϵ.

2. Return to Kaluza–Klein

This section intends to show how to reobtain the Kaluza–Klein structure (P, ω, π, M, g) from the map $\epsilon : N(P) \to E(M)$ when it defines a $\mathbb{C}^*$-principal bundle.

The first problem is to recognize the points of P in $N(P)$; each point p' in P gives origin to the family $Q_{p'}$ of all null geodesics in P coming out from p'. The subset $Q_{p'}$ is, in general, a submanifold of $N(P)$ biholomorphic to $(n-1)$-dimensional quadric (cf. [L]). Moreover the normal bundle $NQ_{p'}$ to this submanifold is of a well determined type and there exists a natural isomorphism between the space $T'_{p'}M$ of (holomorphic) tangent vectors in p' and the space $H^0(Q_{p'}, \mathcal{O}(NQ_{p'}))$ of holomorphic sections of the normal bundle. Throughout this isomorphism the null tangent vectors are in correspondence with those sections that vanish somewhere.

Collecting in $N(P)$ all submanifolds biholomorphic to a $(n-1)$-dimensional quadric with that special type of normal bundle, it is obtained a family parametrized by a manifold $\hat{C}$ of dimension $(n+1)$; the action α of $\mathbb{C}^*$ on $N(P)$ moves to an action of $\mathbb{C}^*$ on $\hat{C}$ and defines a fundamental holomorphic vector field A on $\hat{C}$ (given by $A_c = (\partial/\partial z)|_{z=0}\ (\alpha_{e^z}(c))$).

For every $c \in \hat{C}$ the tangent space $T'_c\hat{C}$ is naturally isomorphic to $H^0(Q_c, \mathcal{O}(NQ_c))$ and the subset Ξ_c of all vectors in T'_c corresponding to sections of NQ_c with a zero somewhere is a cone projecting a regular quadric to the infinity exactly as a set of "null vectors" in T'_c should be (cf. [L, III.2]).

Discarding those points $c \in \hat{C}$ where A_c is in Ξ_c we get an open subset C of $\hat{C}$. In the interesting cases the manifold C contains P as an open subset and it has the same fundamental field A and the same null vectors.

The field A and the null vector cones Ξ_c suffice to reconstruct the metric tensor g' and the connection 1-form ω on P; in fact g' is the only $\mathbb{C}$-bilinear symmetric form on T'_c having Ξ_c as set of null vectors with $g'(A, A) = -k^2$ and ω is given by $\omega(X') = -1/k^2 \cdot g'(A, X')$.

The space of the orbits of C by the action of $\mathbb{C}^*$ is a manifold R of dimension n containing, in the interesting cases, the manifold M. The metric on M and the 2-form Ω are obtained respectively projecting the metric g' and considering the curvature form of the principal bundle $C \to R$.

In the proof of the following theorem some properties from the theory of deformations of complex manifolds will be needed, that can be so summarized (cf. [K,KS]).

Let Q_0 be a compact complex submanifold of a manifold F whose normal bundle NQ_0 satisfies the condition $H^1(Q_0, \mathcal{O}(NQ_0)) = 0$, then there exists a (Kodaira) manifold $C = C(Q_0)$ (the manifold parametrizing the deformations of Q_0) of complex dimension $d = \dim_{\mathbb{C}} H^0(Q_0, \mathcal{O}(NQ_0))$ with a distinguished element c_0 and a submanifold $\mathcal{K}$ of $F \times C$ (the total manifold of the deformation) such that the projection map $pr_2 : \mathcal{K} \to C$ is a regular proper surjection and $Q_0 = pr_1[\mathcal{K} \cap (F \times \{c_0\})]$.

Denoted by $pr_1 : \mathcal{K} \to F$ the projection on the first factor, this implies that as c varies in C the subsets $Q_c = pr_1[\mathcal{K} \cap (F \times \{c\})] = pr_1(\mathcal{K}_c)$ are compact complex submanifolds of F "deforming" the complex structure of Q_0 (it can be proved that all these manifold are isomorphic from the real differentiable viewpoint but not, in general, from the complex differentiable (holomorphic) viewpoint).

For every $c \in C$ the normal bundle of $\mathcal{K}_c$ in $\mathcal{K}$ is a trivial bundle on a compact manifold therefore the map $pr_{2*} : H^0(\mathcal{K}_c, \mathcal{O}(N\mathcal{K}_c)) \to T'_c$ is well defined and a linear isomorphism.

The map $pr_{1*} : H^0(\mathcal{K}_c, \mathcal{O}(N\mathcal{K}_c)) \to H^0(Q_c, \mathcal{O}(NQ_c))$ instead is not, in general, an isomorphism; however under the condition $H^1(Q_0, \mathcal{O}(NQ_0)) = 0$ this is true and for every c in the Kodaira manifold C the map $pr_{1*} \circ pr_{2*}^{-1} : T'_cC \to H^0(Q_c, \mathcal{O}(NQ_c))$ is a natural isomorphism (cf. [K]).

If moreover the conditions $H^1(Q_0, \mathcal{O}(T'Q_0)) = 0$ and $H^1(Q_0, \mathcal{O}(NQ_0 \otimes NQ_0^*)) = 0$ hold then Q_0 and its normal bundle are locally "rigid", that is for every c in an open subset $C'(Q_0)$ of $C(Q_0)$ (containing c_0) the couple (Q_c, NQ_c) is isomorphic to the couple (Q_0, NQ_0) (cf. [KS]).

Given a compact complex manifold K and a holomorphic vector bundle V on K, if the three conditions: $H^1(K, \mathcal{O}(V)) = 0$, $H^1(K, \mathcal{O}(T'K)) = 0$ and $H^1(K, \mathcal{O}(V \otimes V^*)) = 0$ hold and in F there is a submanifold Q_0 such that (Q_0, NQ_0) is isomorphic to the couple (K, V) it is possible to define the space,

$$\hat{C}(K, V) = \{c \in C(Q_0): (Q_c, NQ_c) \text{ is isomorphic to } (K, V)\},$$

of all compact submanifolds of F biholomorphic to K with prescribed normal bundle isomorphic to V.

The space $\hat{C}(K, V)$ is equal to the space $C'(Q_0)$ given above and is therefore a complex manifold of dimension $d = \dim H^0(K, \mathcal{O}(V))$.

The regular m-dimensional quadric Q^m in $\mathbb{P}^{m+1}$ and the bundle $T'\mathbb{P}^{m+1} \mid_{Q^m} \otimes H^*$ verify the three conditions given above (cf. [L, III.2]) and $\dim H^0(Q^m, \mathcal{O}(T'\mathbb{P}^{m+1} \mid_{Q^m} \otimes H^*)) = m + 2$ (for $m \geq 2$) therefore for every complex manifold F it is possible to define the manifold $\hat{C}(m, F)$ of all m-quadrics in F with normal bundle prescribed as $T'\mathbb{P}^{m+1} \mid_{Q^m} \otimes H^*$, the manifold $\hat{C}(m, F)$ (possibly empty) is a complex manifold of dimension $m + 2$ and for every $c \in \hat{C}(m, F)$ there is a natural isomorphism between T'_c and $H^0(Q_c, \mathcal{O}(NQ_c))$.

From now on it will be always supposed $m \geq 2$.

Definition 2.1. Let B be a complex manifold of even (complex) dimension $2 \cdot m$ and let $\beta : F \to B$ a holomorphic principal $\mathbb{C}^*$-bundle on B. A *normal m-quadric in* F is a holomorphically embedded m-quadric with normal bundle isomorphic to $T'\mathbb{P}^{m+1} \mid_{Q^m} \otimes H^*$; the normal quadric is *transverse* if the fibres of F are never tangent to the quadric. An immersed m-quadric of B is called *F-normal* if it is the image on B of a transverse normal m-quadric of F.

Theorem 2.2. *Let B a complex manifold of (complex) dimension $2 \cdot m$ (with $m \geq 2$) and let $\beta : F \to B$ be a holomorphic principal $\mathbb{C}^*$-bundle on B; the space $C = C(B, F)$ of all transverse normal quadric of F is a complex manifold of dimension $m + 2$, the space $R = R(B, F)$ of all F-normal quadrics of B is a complex manifold of dimension $m + 1$. The map $\rho : C \to R$ sending the quadrics of F to quadrics of B makes C a holomorphic principal $\mathbb{C}^*$-bundle on R. The manifold C has a natural holomorphic connection 1-form $\hat{\omega}$ and the manifold R has a natural structure of a holomorphic Riemannian metric $\hat{g}$. That is the 5-tuple $(C, \rho, R, \hat{g}, \hat{\omega})$ is a Kaluza–Klein structure.*

Proof. Let us denote by $\hat{C}$ the $(m+2)$-dimensional manifold for all normal m-quadrics $Q_{p'}$ of F; the holomorphic tangent space $T'_{p'}\hat{C}$ to a point p' of $\hat{C}$ is isomorphic to $H^0(Q_{p'}, \mathcal{O}(NQ_{p'}))$, where $NQ_{p'}$ is the normal bundle of $Q_{p'}$ in F (see the remarks above). The space $\hat{C}$ has a natural conformal structure Ξ (see the remarks above) given assigning for every p' in $\hat{C}$ the cone in $T'_{p'}\hat{C}$ corresponding to the set $\{s \in H^0(Q_{p'}, \mathcal{O}(NQ_{p'})) : s \text{ has a zero }\}$.

The action α of $\mathbb{C}^*$ on F induces an action α on $\hat{C}$; since for every compact subset K of F the subset $X(K) = \{u \in \mathbb{C}^*\colon \alpha_u(K) \cap K \neq \emptyset\}$ is compact in $\mathbb{C}^*$, for every compact subset H of $\hat{C}$ the subset $W(H) = \{u \in \mathbb{C}^*\colon \alpha_u(H) \cap H \neq \emptyset\}$ is also compact in $\mathbb{C}^*$. Therefore for every p' in $\hat{C}$ the group $\{u \in \mathbb{C}^*\colon \alpha_u(p') = p'\}$ is the finite group of $e(p')$th roots of 1 in $\mathbb{C}^*$ for a certain positive natural number $e(p')$ which is locally constant and then constant on the connected components of $\hat{C}$. We will consider on $\hat{C}$ a new action defined by $u \cdot p' = \alpha_v(p')$ where $v^{e(p')} = u$; this action is well defined and moreover has the property $u \cdot p' = p'$ only for $u = 1$.

Denoting by A the fundamental field on $\hat{C}$ defined by the action, let us denote by $C = C(B, F)$ the open subset of $\hat{C}$ of transverse normal quadrics of F, this is exactly the set where A is not a null vector. We are going to prove that on C there is a (unique) holomorphic metric g' defining the conformal structure Ξ, invariant by the action and with quadratic norm $-k^2$ (where $k = m_0/e_0$ is the fixed constant) on all the vectors of the field A. In fact since A is never zero on C for every point p'_0 in C it is possible to find a local chart (V', ϕ') at p'_0 and a polydisc $A^{m+1} \times B^1$ of $\mathbb{C}^{m+2}$ where the action by u becomes the product by u in the last coordinate and where moreover Ξ is defined by a holomorphic metric g''. There is a holomorphic function $f(u, p')$ in a neighborhood of $(1, p'_0)$ such that $\exp(f(u, p')) \cdot g''_{p'}$ gives g'' in $u \cdot p'$ transferred in p' via the action by u; taken a function $h(p')$ such that $A_{p'}(h) = (\partial f/\partial u)(1, p')$ the new metric tensor g' defined by $g'_{p'} = \exp(-h(p')) \cdot g''_{p'}$ verifies the equality $L_A(g') = 0$ (its coefficients do not depend on the last coordinate).

Normalizing g' on the field A we get a metric g' with the desired properties and that is the unique one in its conformal class. The unicity implies that these local metrics patch together giving a global metric on C. In C it is therefore possible to find a base $\mathcal{V}' = \{V'_j\}_{j \in J}$ of the above coordinate open subsets where moreover two points are in the same orbit of the action if and only if they have equal all the coordinates except, at most, the last one. Otherwise near a point p'_0 of C would be possible to find two sequences of points $\{p'_n\}, \{p''_n\}$ converging to p'_0 with $p''_n = u(n) \cdot p'_n$ but such that not all the first $m+1$ coordinates of p'_n and p''_n are equal. Since in a suitable neighborhood of p'_0 for $|u - 1| < \epsilon$ the points p' and $u \cdot p'$ have equal all the first $m+1$ coordinates, it should be definitively $|u_n - 1| \geq \epsilon > 0$, moreover the sequence $\{u_n\}$ contained in the compact subset $W(\{p'_0\} \cup \{p'_n, p''_n\})$ would have a subsequence converging to u_0; but this would imply $u_0 \cdot p'_0 = p'_0$ with $u_0 \neq 1$.

Let us consider now the quotient space R' of all orbits of C via the action of $\mathbb{C}^*$ and the quotient map $\rho : C \to R'$. The map $\sigma : R' \to R$ defined by $\sigma([Q']) = \beta(Q')$ is well defined and surjective. It is also injective because $\beta(Q') = \beta(Q'')$ implies there is a complex non-zero number u such that $Q'' = u \cdot Q'$. In fact the set $X = \{v \in \mathbb{C}^*\colon \alpha_v(Q') \cap Q'' \neq \emptyset\}$ is an analytic subset of $\mathbb{C}^*$ and therefore a finite set $X = \{v_1, \ldots, v_r\}$; for at least one (and only for one) of these complex numbers v_j the intersection $Q'' \cap \alpha_{v_j}(Q')$ has a nonempty interior, for that number we have $Q'' = \alpha_{v_j}(Q')$. We will therefore identify the space $R(B, F)$ with R' via the map σ.

The space R' is a Hausdorff topological space: in fact if $\{p'_n\}, \{p''_n\}$ are nets of equivalent points of C with $p''_n = u(n) \cdot p'_n$ and converging respectively to p', p'' we have $\{u(n)\} \subset W(\{p'_n, p''_n\} \cup \{p', p''\})$ and therefore $p'' = u \cdot p'$ for a non-zero complex number u limit point for the net $\{u(n)\}$. For every chart (V', ϕ') in $\mathcal{V}'$ let us define $V = \rho(V')$ and $\phi : V \to \mathbb{C}^{m+1}$ by $\phi(\rho(p')) = (z^1(p'), \ldots, z^{m+1}(p'))$. The map ϕ is a well-defined homeomorphism between V and an open subset of $\mathbb{C}^{m+1}$. The family $\mathcal{V} = \{V_j\}_{j \in J}$ is a holomorphic atlas making R' a complex manifold of dimension $m+1$.

For every p' in C the set $H_{p'} = \{X' \in T'_{p'}C : g'(X', A) = 0\}$ is a non-degenerate $(m+1)$-dimensional subspace of $T'_{p'}$, the map $\rho_{*p'} : H_{p'} \to T'_{\rho(p')}$ is an isomorphism and the metric tensor g defined on R' by $g_{\rho(p')}(\rho_*(X'), \rho_*(Y')) = g'_{p'}(X', Y')$ for X', Y' in $H_{p'}$ is a well defined holomorphic metric.

The map $\rho : C \to R'$ makes C a principal $\mathbb{C}^*$-bundle on R'. In fact taken two charts (V', ϕ') and (V, ϕ) as in the above proof, the map $\Phi : \rho^{-1}(V) \to V \times \mathbb{C}^*$ defined by $\Phi(u \cdot p') = (\rho(p'), z^{m+2}(p') \cdot u)$ for p' in V' is well defined and it is a trivialization of C on V. On the bundle $\rho : C \to R'$ the 1-form ω defined by $\omega(X') = -1/k^2 \cdot g'(X', A)$ is a holomorphic connection form such that $g' = \rho^* g - k^2 \cdot \omega \otimes \omega$. □

When $\epsilon : N(P) \to E(M)$ is a holomorphic principal $\mathbb{C}^*$-bundle it is possible to compare the original Kaluza–Klein structure (P, π, M, g, ω) and the one obtained applying the previous theorem: $(C(E, N), \rho, R(E, N), \hat{g}, \hat{\omega})$.

When $Q_c = Q_{p'}$ the spaces $T'_{p'}$ and T'_c are both naturally isomorphic to the space $H^0(Q_{p'}, \mathcal{O}(NQ_{p'}))$, therefore the map $\chi : P \to C(E, N)$ defined by $\chi(p') = c$ if $Q_c = Q_{p'}$ is a holomorphic regular map. If $\chi(p') = \chi(p'')$ then the points p' and p'' have the same null geodesics; two (distinct) such points will be called *totally conjugate*.

Theorem 2.3. *Let (P, π, M, g, ω) be a Kaluza–Klein structure such that*
(1) $\epsilon : N(P) \to E(M)$ is a holomorphic principal $\mathbb{C}^*$-bundle,
(2) P does not have couples of totally conjugate points,
then the map $\chi : P \to C(E, N)$ makes (P, π, M, g, ω) an open Kaluza–Klein substructure of $(C(E, N), \rho, R(E, N), \hat{g}, \hat{\omega})$.

Proof. Since all the structures in P and C can be reconstructed from the fundamental field and the holomorphic conformal structure, it is enough to observe that the map χ_* translates these ingredients in the corresponding ones. There is nothing more to add about the conformal structures since we have defined Ξ_c exactly as the set of tangent vectors in c corresponding to those sections in $H^0(Q_c, \mathcal{O}(NQ_c))$ that have a zero somewhere. Some caution is necessary instead about the actions since we have modified, in the course of the proof of the previous theorem, the natural action on C; however, since we suppose there are not totally conjugate points in P, the action by $u \in \mathbb{C}^*$ on P takes $Q_{p'}$ in itself only for $u = 1$, therefore the action does not really change in $\chi(P)$. □

With some more conditions on P it is possible to obtain moreover that $\chi(P) = C$ (if C is connected).

Theorem 2.4. *Let (P, π, M, g, ω) be a Kaluza–Klein structure such that*
(1) *$\epsilon : N(P) \to E(M)$ is a holomorphic principal $\mathbb{C}^*$-bundle,*
(2) *P does not have couples of totally conjugate points,*
(3) *P is civil,*
(4) *P is Stein,*
(5) *$H^2(P, \mathbb{Z}) = 0$,*
(6) *the null geodesic sets in P are contractible,*
then the map $\chi : P \to C(E, N)$ is an isometry between (P, g') and a connected component of $(C(E, N), \hat{g}')$.

Proof. The added hypotheses are those required by Theorem III.5 in [L] in order to guarantee that the map χ is a conformal biholomorphism between P and the connected component of C containing $\chi(P)$. □

A natural problem in this context is the following: What does B represent for the Kaluza–Klein structure $(C(B, F), \rho, R(B, F), \hat{g}, \hat{\omega})$?

Associated to every element $b \in B$ there is an immersed curve $\gamma_b = \{p \in R{:}\ b \in Q_p\}$. Under a natural condition on the principal bundle $\beta : F \to B$ it is possible to prove that each of these curves is composed by one or more world-lines of particles in the triplet.

The condition required is the presence on the bundle space F of a holomorphic contact structure. The contact structure does not intervene in the construction of the Kaluza–Klein 5-tuple $(C(B, F), \rho, R(B, F), \hat{g}, \hat{\omega})$, but when it is present it guarantees that a certain "conformal connection" induced by F on C coincides with the "natural conformal connection" defined by the conformal structure of C and there are not "conformal torsion tensors" contributing (cf. [L, II.2 and III.3]).

More precisely this means that as b' varies in F the curves $\gamma_{b'} = \{c \in C{:}\ b' \in Q_c\}$ are immersed curves in C whose components are the null geodesics of $(C, \hat{g}')$; without the contact structure on F the curves $\gamma_{b'}$ are, in general, only null curves and (locally) geodesics for a holomorphic connection different from the (holomorphic) Levi-Civita connection of $(C, \hat{g}')$.

Definition 2.5. Let F be a complex manifold, a holomorphic *contact structure* on F is a holomorphic distribution H of complex tangent hyperplanes on F such that the Frobenius bilinear map $\Phi : H \times H \to T'F/H$ (defined by $\Phi(X, Y) = [X, Y] \bmod H$) is non-degenerate.

A distribution H of hyperplanes on F is a contact structure if and only if the manifold F has an odd (complex) dimension $2{\cdot}m + 1$ and H is locally the zero-set of a (holomorphic) 1-form θ such that $\theta \wedge (\partial\theta)^{\wedge m}$ is never zero.

Theorem 2.6. *Let B a complex manifold of (complex) dimension $2 \cdot m$ (with $m \geq 2$) and let $\beta : F \to B$ be a holomorphic principal $\mathbb{C}^*$-bundle on B. When the bundle manifold F has a contact structure the subsets $\gamma_b = \{p \in R\colon b \in Q_p\}$ (as b varies in B) make a family of immersed curves in $R = R(B, F)$ whose components are exactly the supports of the world-lines of the particles in the triplet.*

Proof. The curves γ_b in R correspond in R' to the curves $\hat{\gamma}_b = \{[Q]\colon b \in \beta(Q)\}$ images through ρ of the curves $\gamma_{b'} = \{Q \in C\colon b' \in Q\}$ of C (when $\beta(b') = b$). The curves $\gamma_{b'}$ are, for the Theorems III.2 and III.3 of [L], immersed curves in C whose components are the null geodesics of (C, g') (cf. also the remarks above); therefore, since in C a (non-zero) null vector is never vertical, the $\hat{\gamma}_b$ are immersed curves of R'.

For every null geodesic $\gamma : D \to C$ running along a $\gamma_{b'}$ we want to prove that $\rho \circ \gamma : D \to R'$ is the world-line of a particle (running along $\hat{\gamma}_{\beta(b')}$). Since the problem is local we choose a chart V' in C and a chart V in R' in such a way that $V = \rho(V')$ and $V' \simeq V \times G$ has as coordinates the coordinate functions $z^1, \ldots, z^{m+1}$ of V and moreover a coordinate w on the open region G in $\mathbb{C}$ such that $\partial_w = A$.

The curve $\gamma = (\alpha, \beta) : D \to V \times G$ is stationary with respect to the "length" defined by the "arc-length" $\mathcal{L} : T'C \to \mathbb{C}$ given by $\mathcal{L}(X') = g'(X', X') = \mathcal{L}(z, w, Z, W) = g_{ab}(z) \cdot Z^a \cdot Z^b - k^2 \cdot [\omega_a(z) \cdot Z^a + W]^2$, therefore as a geodesic verifies the Lagrange equations

(1) $\omega \rfloor \dot{\gamma} = \text{constant} = Q,$

(2) $\nabla_{\dot{\alpha}} \dot{\alpha} = Q \cdot (\uparrow \Omega \rfloor \dot{\alpha}),$

and as a null curve

(3) $g(\dot{\alpha}, \dot{\alpha}) - k^2 \cdot Q^2 = 0.$

Changing the affine parameter on γ we can suppose $Q = \sigma \cdot e_0$ (with $\sigma \in \{-1, 0, 1\}$); then $\rho \circ \gamma = \alpha$ is the world-line of a particle in the triplet.

Conversely let $\alpha : D \to V$ be the world-line of a particle with rest-mass and charge $(\sigma^2 \cdot m_0, \sigma \cdot e_0)$ with $\nabla_{\dot{\alpha}} \dot{\alpha} = \sigma \cdot e_0 \cdot (\uparrow \Omega \rfloor \dot{\alpha})$; restricting, in case, D we can find a holomorphic map $\beta : D \to G$ such that $\dot{\beta} = \sigma \cdot e_0 - \omega_a \cdot \dot{\alpha}^a$. The map $\gamma = (\alpha, \beta) : D \to V \times G$ verifies the Lagrange equations (1) and (2) and therefore is a geodesic, moreover $g'(\dot{\gamma}, \dot{\gamma}) = \sigma^4 \cdot m_0^2 - \sigma^2 \cdot k^2 \cdot e_0^2 = 0$. If $\gamma_{b'}$ contains $\gamma(D)$, then $\alpha(D)$ runs along $\hat{\gamma}_{\beta(b')}$. □

When the manifold P is civil with $\dim P \geq 4$ the manifold $N(P)$ has a contact structure H (cf. [L, III.3]). For every $\gamma \in N(P)$ the hyperplane H_γ is generated by the union of tangent linear spaces $T'_\gamma Q_{p'}$ as p' varies in the null geodesic γ. The contact structure H is invariant by the action of $\mathbb{C}^*$ on $N(P)$.

Therefore the theorem above applies to the case $\epsilon : N(P) \to E(M)$ in which we are most interested.

3. Meromorphic symplectic forms

Let B a complex manifold of complex dimension $2 \cdot m$ and let $\beta : F \to B$ be a holomorphic principal $\mathbb{C}^*$-bundle with a holomorphic contact structure H invariant by the action of $\mathbb{C}^*$; on F we denote by A the fundamental holomorphic (never zero, invariant) vector field whose integral curves are the orbits of the action in F.

We can find on B an open covering $\mathcal{U} = \{(U_j, \Phi_j)\}_{j\in J}$ by Stein local charts with images in polydisks of $\mathbb{C}^{2\cdot m}$ and with contractible finite intersections, and on F a similar open covering $\mathcal{V} = \{(V_j, \Psi_j)\}_{j\in J}$ in such a way that the map β looks like the projection on the first $2\cdot m$ coordinates and the action of $\mathbb{C}^*$ on V_j like the multiplicative action on the last coordinate. We can suppose the contact structure H is defined on V_j by the 1-form θ_j and choose for every bundle $\beta : V_j \to U_j$ a holomorphic section $s_j : U_j \to V_j$ with constant value in the last coordinate.

It is possible to prove that on every U_j there is a holomorphic function r_j such that the quotient $(\theta_j \rfloor A)/(r_j \circ \beta)$ is a never zero function on V_j; the 1-forms $\alpha_j = s_j^*[\theta_j/(\theta_j \rfloor A)]$ verify the relation

$$\alpha_k - \alpha_j = \partial[\log(s_k/s_j)],$$

therefore the meromorphic 2-form $\sigma = \partial\alpha_j$, it is well defined on B with poles on the zero-set of the functions (r_j). The form $\sigma = \sigma(F, H, B)$ (independent from the open coverings chosen) will be called the *curvature form of the bundle F on B with contact structure H.*

Let B be a complex manifold and D an effective divisor defined in every point b in B by the germ of the function r_b in $\mathcal{O}_b$. On B we will consider the sheaf $\mathcal{Z}^1$ of germs of closed holomorphic 1-forms, the sheaf $\mathcal{S}_D^1 = (1/r)\cdot\mathcal{O}^1$ of germs of meromorphic 1-forms with "simple poles in D", the image sheaf $\mathcal{T}_D^2 = \partial\mathcal{S}_D^1$ (made of germs of meromorphic 2-forms), the subsheaf $\mathcal{Z}_D^1$ of $\mathcal{S}_D^1$ of closed 1-forms and the quotient sheaf $\mathcal{R}_D^1 = \mathcal{Z}_D^1/\mathcal{Z}^1$ (with support contained in the support of D); all these sheaves are coherent and are connected by the following two exact sequences:

$$0 \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{Z}_D^1 \xrightarrow{q} \mathcal{R}_D^1 \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{Z}_D^1 \longrightarrow \mathcal{S}_D^1 \xrightarrow{\partial} \mathcal{T}_D^2 \longrightarrow 0.$$

In this context we will consider the map $\rho_D : H^0(B, \mathcal{T}_D^2) \to H^1(B, \mathcal{R}_D^1)$ defined by $\rho_D(\sigma) = q^*(\delta(\sigma))$ that gives for every σ a kind of "cohomological residue" of σ along D.

As usual we will denote by (Φ) the divisor defined in $B^{2\cdot m}$ by the meromorphic $2\cdot m$-form Φ.

Definition 3.1. Let B a complex manifold of dimension $2\cdot m$ and S a regular hypersurface of B a *Konstant–Souriau form on B with poles in S* is a meromorphic 2-form σ in $H^0(B, \mathcal{T}_S^2)$ such that

(1) $(\sigma^{\wedge m}) = -(m+1)\cdot S$,

(2) $\sigma|_{B-S}$ extends to an integral class on B,

(3) $\rho_S(\sigma) = 0$.

When $S = \emptyset$ a Konstant–Souriau form reduces to a (holomorphic) symplectic form satisfying the Weil's integrality condition as in Konstant–Souriau theory of geometric quantization (cf. [W]).

Theorem 3.2. *Let B a complex manifold of dimension $2 \cdot m$, a meromorphic 2-form σ on B is the curvature form of a principal $\mathbb{C}^*$-bundle on B furnished with an invariant contact structure if and only if it is a Konstant–Souriau form on B with poles in a regular hypersurface.*

Proof. Let σ be the curvature form of the principal $\mathbb{C}^*$-bundle $\beta : F \to B$ furnished with an invariant contact structure H. Let us consider on B and F two open coverings $\mathcal{U} = \{(U_j, \Phi_j)\}_{j \in J}$ and $\mathcal{V} = \{(V_j, \Psi_j)\}_{j \in J}$ as in the definition of curvature form (cf. the remarks above) and let us suppose H be defined on V_j by the 1-form θ_j, there is, on $U_j \cap U_k$, a holomorphic function t_{jk} with value in $\mathbb{C}^*$ such that $\theta_j = t_{jk} \cdot \theta_k$. We pose $f_j = \theta_j \rfloor A$ and call T the zero-set of the functions (f_j), the 1-form $\hat{\theta} = (1/f_j) \cdot \theta_j$ is a well-defined meromorphic form on F without zeros and with poles in T; moreover: $\hat{\theta} \rfloor A = 1$ on $F - T$ and $\hat{\theta}$ is invariant by the action.

We have $f_j = t_{jk} \cdot f_k$ and on every U_j it is possible to find a function r_j such that $f_j/(r_j \circ \beta)$ is never zero on V_j. Called D the divisor defined by the system (r_j) and S its support, we have $T = \beta^{-1}(S)$. The meromorphic 1-form $\alpha_j = s_j^*(\hat{\theta}) = [1/(f_j \circ s_j)] \cdot s_j^*(\theta_j)$ is in $\mathcal{S}^1_D(U_j)$ without points of indeterminancy and verifies the equations

$$\alpha_k - \alpha_j = \partial[\log(h_{jk})] \quad \text{and} \quad \hat{\theta} = \beta^*(\alpha_j) + \Psi_j^*(\partial \log(u_j)),$$

where h_{jk} is the never zero holomorphic function defined in $U_j \cap U_k$ by the equality $h_{jk}(b) \cdot s_j(b) = s_k(b)$ and u_j is the last coordinate function for the chart (V_j, Ψ_j); as in the definition we take $\sigma = \partial \alpha_j$.

In V_j we have (forgetting, for a moment, about the index j): $(\sigma^{\wedge m}) = (\sigma^{\wedge m} \wedge \partial \log(u)) = (f^{-(m+1)} \cdot \theta \wedge (\partial\theta)^{\wedge m}) = -(m+1) \cdot D$; the 1-form $r \cdot \alpha$ is holomorphic in U_j and since $r \cdot [\partial(r \cdot \alpha)]^{\wedge m} - m \cdot \partial r \wedge (r \cdot \alpha) \wedge [\partial(r \cdot \alpha)]^{\wedge (m-1)} = r^{m+1} \cdot \sigma^{\wedge m}$ is never zero it must be $\partial r_j \neq 0$ when $r_j = 0$, therefore S is regular hypersurface and $D = 1 \cdot S$.

The cocycle (c_{jkl}) with $c_{jkl} = (1/2\pi \mathrm{i}) \cdot [\log h_{jk} + \log h_{kl} + \log h_{lj}]$ defines an integral class in $H^2(B, \mathbb{C})$ whose restriction to $B - S$ is the class of the closed 2-form σ.

At last $\rho_S(\sigma) = q^*[\alpha_k - \alpha_j] = [\partial \log(h_{jk}) \bmod \mathcal{Z}^1] = 0$.

Conversely let σ be a Konstant–Souriau form with poles on the regular hypersurface S of B and let $\mathcal{U}$ be an open covering of B with the usual properties and such that on every U_j the ideal sheaf of S is defined by the function r_j and σ is expressible as $\sigma = \partial \beta_j$ with β_j in $\mathcal{S}^1_S(U_j)$.

The condition $\rho_S(\sigma) = 0$ implies that for every j there is a 1-form μ_j in $\mathcal{Z}^1_S(U_j)$ and for every couple j, k there is a holomorphic function r_{jk} on $U_j \cap U_k$ such that $(\beta_k - \beta_j) - (\mu_k - \mu_j) = \partial r_{jk}$.

Let us define on every U_j the form $\alpha_j = \beta_j - \mu_j$, this is a form in $\mathcal{S}^1_S(U_j)$ with $\partial\alpha_j = \sigma$ and $\alpha_k - \alpha_j = \partial[\log(\exp(r_{jk}))]$; the cocycle $(r_{jk} + r_{kl} + r_{lj})$ is made of constant functions and restricted to $B - S$ defines the class of σ in $H^2(B - S, \mathbb{C})$. Since this extends to an integral class in B we have $\exp[r_{jk} + r_{kl} + r_{lj}] = 1$ in $H^2(B, \mathbb{C}^*)$; there is a 1-cochain (b_{jk}) of non-zero complex numbers such that $\exp(r_{jk} + r_{kl} + r_{lj}) = b_{jk} \cdot b_{kl} \cdot b_{lj}$ on $U_j \cap U_k \cap U_l$ and therefore if we pose $g_{jk} = \exp(r_{jk})/b_{jk}$ we get

$$\alpha_k - \alpha_j = \partial[\log(g_{jk})] \quad \text{and} \quad g_{jk} \cdot g_{kl} \cdot g_{lj} = 1.$$

Moreover the form α_j does not have points of indeterminancy, in fact (forgetting, for a moment, about the index j) since the form $r^{m+1} \cdot \sigma^{\wedge m} = r \cdot [\partial(r \cdot \alpha)]^{\wedge m} - m \cdot \partial r \wedge (r \cdot \alpha) \wedge [\partial(r \cdot \alpha)]^{\wedge(m-1)}$ is never zero it must be $(r \cdot \alpha)_b \neq 0$ when $r(b) = 0$.

Let us consider the principal $\mathbb{C}^*$-bundle $\beta : F \to B$ defined by the transition functions (g_{jk}), the open charts $V_j = \beta^{-1}(U_j)$ and the natural biholomorphism $\epsilon_j : V_j \to U_j \times \mathbb{C}^*$. The 1-form $\theta_j = \epsilon_j^*[r_j \cdot \alpha_j + r_j \cdot \partial \log(u_j)]$, where u_j is the last coordinate in $U_j \times \mathbb{C}^*$, is holomorphic in V_j and never zero; these different forms are related on the intersections by the equations $\theta_j = (r_j/r_k) \circ \beta \cdot \theta_k$ and therefore for every point b' in F it is well defined a linear hyperplane $H_{b'}$ in $T'_{b'}F$ as the kernel of any of the forms θ_j defined at b'.

The distribution $(H_{b'})_{b' \in F}$ is a contact structure on F because $\theta_j \wedge (\partial\theta_j)^{\wedge m} = r^{m+1} \cdot \partial[\log(u_j)] \wedge \sigma^{\wedge m}$ is never zero. □

When the manifold P is civil with $\dim P \geq 4$ and $\epsilon : N(P) \to E(M)$ is a holomorphic principal $\mathbb{C}^*$-bundle the theorem above applies and proves that $E(M)$ is furnished with a Konstant–Souriau meromorphic symplectic form σ with poles in a regular hypersurface S of $E(M)$.

Denoted by $E_0(M)$ the subset made of all null geodesics of (M, g) it is not difficult to prove that $E_0(M) \subset S$.

In fact if α is a null geodesic in (M, g), taken a point p in the support of α and a null vector V in p tangent to the geodesic it is possible to find a point p' in P and a null vector W in p' projecting on V and such that the fundamental vector $A_{p'}$ is tangent in W to the null tangent cone $\Xi_{p'}$ in p'. This implies there is a holomorphic curve $\xi : D \to \Xi_{p'}$ with $\xi(0) = W$ and $\dot{\xi}(0) = A_{p'}$; the map ξ induces a map $\hat{\xi} : D \to N(P)$ such that $\hat{\xi}(0) = \gamma$ (a null geodesic of P that projects on α) and $\dot{\hat{\xi}}(0) = A_\gamma$. In the same time $\dot{\hat{\xi}}(0)$ is in $T'_\gamma Q_{p'}$ and therefore the null geodesic γ belongs to the set T of the previous proof, then α belongs to S.

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